PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Stability of Flow in a Rotating Viscous Incompressible Fluid Subjected to Differential Heating

J. Brindley

Phil. Trans. R. Soc. Lond. A 1960 **253**, 1-25
doi: 10.1098/rsta.1960.0016

STABILITY OF FLOW IN A ROTATING VISCOUS INCOMPRESSIBLE FLUID SUBJECTED TO DIFFERENTIAL HEATING

By J. BRINDLEY

Department of Mathematics, University of Leeds

(*Communicated by T. G. Cowling, F.R.S.—Received* 14 *October* 1959)

CONTENTS

		PAGE			PAGE
	Introduction	1	5.	Derivation of a solution for f and comparison with experiment	13
1.	Formulation of the problem	2		Appendix A	19
2.	Solution of perturbation equations	5		Appendix B	21
3.	Solution of the equation for $\overline{W}$	7		References	25
4.	The boundary conditions	10			

An attempt is made to find a theoretical explanation for the type of flow observed when a liquid is subjected at the same time to rotation and to a horizontal temperature gradient. When the liquid is contained between two concentric cylinders it is known from experiment that two distinct types of flow occur, one in which the motion is in the form of a meridional vortex with the addition of a zonal component, and the other in which the motion exhibits a meandering wave-like pattern. Of vital importance as regards the type of flow is found to be a parameter defined as the Rossby number for the problem, and two sets of critical values of this parameter are found which bound the range over which wave motion is possible. Qualitatively this is in complete agreement with experimental observations, but quantitative results show some discrepancy between theory and experiment.

Introduction

Experiments by Fultz at the University of Chicago and Hide at Cambridge on the motion of liquids caused to rotate and subjected to differential heating, as described by Fultz (1956) and Hide (1958), and the subsequent recognition of the importance of such experiments in the field of meteorology, have inspired a number of theoretical investigations of the dynamics and thermodynamics of the motions produced. In particular, Kuo (1954, 1955, 1956*a*, *b*), Davies (1953, 1956, 1959) and Lorenz (1953) have formulated the problem mathematically and have established criteria for the onset of instability. The present paper consists principally of an extension of the methods used by Davies (1956).

The experiments themselves have taken two slightly different forms; originally Fultz carried out a series of so-called 'dishpan' experiments, in which a rotating flat vessel, 30 cm in diameter and containing water to a depth of about 4 cm, was heated from below by an annular heating element situated close to the rim. Subsequently he used an apparatus similar to that of Hide, consisting of two concentric cylinders, maintained at constant but differing temperatures, containing between them the experimental liquid. The dimensions in this second case were somewhat different in that the depth of the water, about 10 cm, equalled the diameter of the outer cylinder. Nevertheless, the phenomena observed in the two cases were very similar, two distinct régimes of flow being recognized. At low rotation speeds the flow was of the type usually known as the Hadley cell, or smoke-ring type, i.e. a meridional vortex in which the motion perpendicular to the axis of rotation is deflected by Coriolis forces, thus giving rise to a zonal component of velocity. Hadley suggested in the eighteenth century that such a circulation could account for the trade winds. At higher rotation speeds a completely new type of flow set in, distinguished by the presence of a narrow stream of liquid travelling with relatively high velocity in the direction of rotation and meandering in a wave pattern, rather irregular in the dishpan case, but extremely regular and persistent in the experiments with two concentric cylinders. This stream appears to bear a qualitative resemblance to the atmospheric 'jet-stream', which is a conspicuous feature of the middle latitude circulation at high levels.

Both Hide and Fultz have ascribed critical importance, as regards the characteristics of the flow, to a single parameter which has come to be known as a Rossby number, and which is essentially of the form

$$R_H \equiv \frac{g\alpha d(\Delta T)_H}{\rho r^2 \Omega^2},$$

where d = depth of liquid,
ρ = mean density,
r = horizontal dimension of apparatus,
Ω = angular velocity of apparatus,
$(\Delta T)_H$ = imposed horizontal temperature contrast,
α = coefficient of expansion of liquid.

They have shown that for values of R_H greater than some critical value, $R_{H,\text{crit.}}$, the flow is essentially spiral, while for $R_H < R_{H,\text{crit.}}$ the flow is of a wave-like pattern, motion being largely confined to the meandering jet-stream. The wave-number in the second case depends on R_H, and increases as R_H decreases within certain limits which appear to be imposed by the geometry of the system. Recent experiments of Fultz, in which he has observed symmetric flow patterns for much lower values of R_H than hitherto, suggest that a symmetric flow pattern should be theoretically possible for small R_H. This is in agreement with the results of the present work, which seem to indicate that, below a certain value of R_H, motion of the wave type will be damped out by viscous effects.

1. Formulation of the problem

In the mathematical formulation of the problem, cylindrical co-ordinates (r, ϕ, z) are used, and there are six dependent variables $(u_1, v_1, w_1, p_1, \rho_1, T_1)$, where u_1, v_1, w_1 represent the velocity components in the directions r increasng, ϕ increasing, and z increasing, respectively, and p_1 is the pressure, ρ_1 the density, and T_1 the temperature. These six dependent variables are connected by the following six equations:

$$\rho_1\left(\frac{\mathrm{d}u_1}{\mathrm{d}t}-\frac{v_1^2}{r}\right) = -\frac{\partial p_1}{\partial r}+\mu\left(\nabla^2 u_1-\frac{u_1}{r^2}-\frac{2}{r^2}\frac{\partial v_1}{\partial \phi}\right), \tag{1·1}$$

$$\rho_1\left(\frac{\mathrm{d}v_1}{\mathrm{d}t}+\frac{u_1 v_1}{r}\right) = -\frac{\partial p_1}{r\partial \phi}+\mu\left(\nabla^2 v_1-\frac{v_1}{r^2}+\frac{2}{r^2}\frac{\partial u_1}{\partial \phi}\right), \tag{1·2}$$

$$\rho_1\frac{\mathrm{d}w_1}{\mathrm{d}t} = -\frac{\partial p_1}{\partial z}-g\rho_1+\mu\nabla^2 w_1, \tag{1·3}$$

$$\rho_1 c_v\frac{\mathrm{d}T_1}{\mathrm{d}t} = k\nabla^2 T_1, \tag{1·4}$$

$$\rho_1-\rho_0 = \alpha(T_1-T_0), \tag{1·5}$$

$$\frac{\partial u_1}{\partial r}+\frac{u_1}{r}+\frac{\partial v_1}{r\partial \phi}+\frac{\partial w_1}{\partial z} = 0. \tag{1·6}$$

Equations (1·1) to (1·3) are the equations of motion; (1·4) is the equation of heat transfer for a liquid, generation of heat by molecular viscosity being neglected; (1·5) is the equation of state, and (1·6) the equation of continuity. The slight deviation of the density ρ_1 from a constant value ρ_0 is ignored everywhere except in the buoyancy term $g\rho_1$ in equation (1·3). In this term the linear relation (1·5) is used; this is sufficiently accurate within the temperature range of the experiments, usually about 10 to 25 °C.

The experiments have revealed that, when the temperature field is averaged for all ϕ, the resulting mean distribution shows an increase in temperature both from the base to the free surface, and from the central axis (or inner cylindrical wall) to the outer cylindrical wall. It is evidently possible to regard the temperature pattern in any ϕ-plane as a small departure from this mean temperature field, and accordingly we write

$$\rho_1 = \rho^*(r,z) + \rho(r,z,\phi,t), \tag{1·7}$$

$$T_1 = T^*(r,z) + T(r,z,\phi,t), \tag{1·8}$$

where ρ^* and T^* represent the mean meridional fields of density and temperature, respectively, and are functions of (r,z) only, while ρ, T represent the small departures from these mean fields.

As a result of the mean temperature field, there exists a mean zonal motion in the direction of ϕ increasing, relative to the rotating cylinder. We therefore assume that the complete velocity field can be represented in the form

$$u_1 = u(r,z,\phi,t), \quad v_1 = r\Omega + V^*(r,z) + v(r,z,\phi,t), \quad w_1 = w(r,z,\phi,t), \tag{1·9}$$

where $V^*(r,z)$ is the mean relative zonal velocity associated with the mean temperature field. The expression for the pressure is

$$p_1 = P^*(r,z) + p(r,z,\phi,t), \tag{1·10}$$

where $P^*(r,z)$ is the mean pressure associated with the mean zonal flow and hydrostatic sources. We are now able to write down the equations governing the mean zonal motion as follows

$$\left.\begin{aligned} -\frac{\rho_0(r\Omega+V^*)^2}{r} &= -\frac{\partial P^*}{\partial r}, && (a)\\ 0 &= \nabla^2 V^* - V^*/r^2, && (b)\\ 0 &= -\partial P^*/\partial z - g\rho^*, && (c)\\ \rho^* &= \rho_0 - \alpha(T^* - T_0), && (d)\\ 0 &= k\nabla^2 T^*. && (e)\end{aligned}\right\} \tag{1·11}$$

The use of an exact solution of the heat-transfer equation (1·11 e) leads to considerable difficulties in the solution of the perturbation equations later. We therefore follow Davies (1956) in replacing this equation by

$$T^* = T_0 + \Theta_v z + \Theta_H \tfrac{1}{2} r^2, \tag{1·12}$$

where Θ_v and Θ_H are positive constants. This in fact represents a temperature field which could only be maintained by sinks of heat in the fluid. Ignoring the term $\rho_0 V^{*2}/r$ on the left-hand side of (1·11 a), on the ground that V^* is considerably less than $r\Omega$, we find that the corresponding solution for V^* is

$$V^* = 2\Omega R_H (z/h)\, r. \tag{1·13}$$

Here h is the depth of the liquid and R_H is a non-dimensional constant given by

$$R_H = \frac{g\alpha h\Theta_H}{4\rho_0\Omega^2}, \tag{1·14}$$

which is at once identifiable with the parameter defined by Fultz and Hide, and mentioned above. The velocity expression (1·13) clearly cannot be valid for all values of r, since the velocity V^* must vanish at the inner and outer cylindrical walls, but it may be used as an approximation outside the boundary layers.

Before the equations governing the perturbation field are written down, one other approximation must be described. In the equation of heat transfer (1·4) the left-hand side includes a term

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t}+\frac{V^*+v}{r}\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)(T^*+T)$$

$$\equiv \frac{\partial T}{\partial t}+\frac{V^*}{r}\frac{\partial T}{\partial \phi}+u\frac{\partial T^*}{\partial r}+w\frac{\partial T^*}{\partial z}+\text{second-order terms},$$

where ϕ is measured relative to a radius fixed in the cylinder. Direct use of equation (1·12) would then lead to the replacement of $\partial T^*/\partial z$ by Θ_v and $\partial T^*/\partial r$ by $r\Theta_H$. However, the use of this operator in the heat transfer equation would mean that the method of separation of variables could not be applied to the perturbation equations. For this reason ru will be replaced in this operator by an approximate value obtained from equation (1·2) by considering only the leading terms, namely the Coriolis term $2\Omega u$, and the pressure gradient $\partial p/r\partial\phi$. That is, ru is replaced by $-(1/2\Omega\rho_0)(\partial p/\partial\phi)$; accordingly the expression for $\mathrm{d}T/\mathrm{d}t$ is written in the form

$$\frac{\mathrm{d}T}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t}+\frac{V^*}{r}\frac{\partial}{\partial\phi}\right)T+\Theta_v w-\frac{\Theta_H}{2\Omega\rho_0}\frac{\partial p}{\partial\phi}+\text{second-order terms}. \tag{1·15}$$

The equations connecting the perturbation quantities are now assumed to be the linearized form of equations (1·1) to (1·6), which, including the above approximations, are

$$\rho_0\left\{\left(\frac{\partial}{\partial t}+\frac{V^*}{r}\frac{\partial}{\partial\phi}\right)u-\left(2\Omega+\frac{2V^*}{r}\right)v\right\} = -\frac{\partial p}{\partial r}+\mu\left\{\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\phi}\right\}, \tag{1·16}$$

$$\rho_0\left\{\left(\frac{\partial}{\partial t}+\frac{V^*}{r}\frac{\partial}{\partial\phi}\right)v+\left(2\Omega+\frac{V^*}{r}+\frac{\partial V^*}{\partial r}\right)u\right\} = -\frac{\partial p}{r\partial\phi}+\mu\left\{\nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial\phi}\right\}, \tag{1·17}$$

$$\rho_0\left(\frac{\partial}{\partial t}+\frac{V^*}{r}\frac{\partial}{\partial\phi}\right)w = -\frac{\partial p}{\partial z}+g\alpha T+\mu\nabla^2 w, \tag{1·18}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{r\partial\phi}+\frac{\partial w}{\partial z} = 0, \tag{1·19}$$

$$\rho_0 c_v\left\{\left(\frac{\partial}{\partial t}+\frac{V^*}{r}\frac{\partial}{\partial\phi}\right)T-\frac{\Theta_H}{2\Omega\rho_0}\frac{\partial p}{\partial\phi}+\Theta_v w\right\} = k\nabla^2 T. \tag{1·20}$$

The justification of this linearization is that the problem under review is essentially one of stability, most interest being centred on the behaviour of the initial small departures from the mean flow. The first equations (1·16) to (1·20) form a consistent set connecting the five dependent variables u, v, w, p, T, and it remains to find a suitable form of solution.

2. Solution of perturbation equations

We first of all assume that the dependent variables u, v, w, p, T each depend on ϕ and t through a factor $\exp(\mathrm{i}m\phi+\mathrm{i}\sigma t)$, where m is the wave-number of the perturbation and σ its frequency. Also, in order to employ the method of separation of variables we further follow Davies (1956) in introducing new variables $\xi(z)$, $\eta(z)$, $W(z)$, $P(z)$, $\tau(z)$ defined by

$$\left.\begin{aligned} 2u &= \xi(z)\,\frac{mC_m(\beta r)}{\beta r}-\eta(z)\,C'_m(\beta r), && (a)\\ 2\mathrm{i}v &= -\xi(z)\,C'_m(\beta r)+\eta(z)\,\frac{mC_m(\beta r)}{\beta r}, && (b)\\ w &= W(z)\,C_m(\beta r), && (c)\\ p &= P(z)\,C_m(\beta r), && (d)\\ T &= \tau(z)\,C_m(\beta r), && (e)\end{aligned}\right\} \tag{2·1}$$

where $C_m(\omega)$ is a solution of Bessel's equation of order m

$$\frac{\mathrm{d}^2C_m}{\mathrm{d}\omega^2}+\frac{1}{\omega}\frac{\mathrm{d}C_m}{\mathrm{d}\omega}+\left(1-\frac{m^2}{\omega^2}\right)C_m = 0. \tag{2·2}$$

This substitution leads to the following system of five ordinary simultaneous equations.

$$\mathrm{i}\rho_0\sigma'\xi+2\mathrm{i}\rho_0\Omega'\eta = \mu\left(\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2}-\beta^2\xi\right), \tag{2·3}$$

$$\mathrm{i}\rho_0\sigma'\eta+2\mathrm{i}\rho_0\Omega'\xi = 2\beta P+\mu\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2}-\beta^2\eta\right), \tag{2·4}$$

$$\mathrm{i}\rho_0\sigma' W-g\alpha\tau = -\frac{\mathrm{d}P}{\mathrm{d}z}+\mu\left(\frac{\mathrm{d}^2W}{\mathrm{d}z^2}-\beta^2W\right), \tag{2·5}$$

$$\beta\eta+2\frac{\mathrm{d}W}{\mathrm{d}z} = 0, \tag{2·6}$$

$$c_v\rho_0\left\{\mathrm{i}\sigma'\tau-\frac{\mathrm{i}m\Theta_H P}{2\rho_0\Omega}+\Theta_v W\right\} = k\left\{\frac{\mathrm{d}^2\tau}{\mathrm{d}z^2}-\beta^2\tau\right\}. \tag{2·7}$$

Here σ', Ω' are defined by

$$\sigma' = \sigma+2\Omega R_H mz/h, \quad \Omega' = \Omega(1+2R_H z/h); \tag{2·8}$$

they may be regarded as the effective frequency and effective angular velocity. Their dependence on the vertical height z is a direct consequence of the baroclinicity, or non-coincidence of isobaric and isothermal surfaces in the unperturbed state.

The function $C_m(\beta r)$ introduced in equations (2·1) is any permissible solution of equation (2·2); for a liquid bounded by one outer cylindrical wall $C_m(\beta r)$ must be taken as $J_m(\beta r)$, whereas for a liquid contained between two cylindrical walls we must have

$$C_m(\beta r) = \alpha_m J_m(\beta r)+\beta_m Y_m(\beta r).$$

The equations (2·4) to (2·8) are identical in form in the two cases. This agrees with the experimental result of Fultz and Hide that the stability characteristics of the two cases are not essentially different, for these characteristics depend only on the parameters occurring in equations (2·4) to (2·8) and on the boundary conditions at the base and at the surface of the liquid.

The exact boundary conditions at the outer wall $r = r_o$ are $u = v = w = 0$. However, since the equations to be used are not themselves valid in the boundary layer next to the wall, it is sufficient merely to require the vanishing of the normal component of the velocity at the wall, i.e.

$$u = 0 \quad \text{at} \quad r = r_o. \tag{2·9}$$

In the case of liquid contained between two cylinders there is evidently a second boundary condition of this type at the inner boundary $r = r_i$.

At the base of the liquid, $z = 0$, and at the free surface, $z = h$, the boundary conditions are

$$u = v = w = 0 \quad \text{at} \quad z = 0,$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at} \quad z = h.$$

These are equivalent to

$$\xi = \eta = W = 0 \quad \text{at} \quad z = 0, \tag{2·10}$$

$$\frac{d\xi}{dz} = \frac{d\eta}{dz} = W = 0 \quad \text{at} \quad z = h. \tag{2·11}$$

In addition, conditions have to be imposed on the heat flow across these surfaces. It is assumed that there is no perturbation heat flow across either surface. Thus

$$d\tau/dz = 0 \quad \text{at} \quad z = 0 \quad \text{and at} \quad z = h. \tag{2·12}$$

It is convenient to replace the variables ξ, η, W, P, τ and z in equations (2·4) to (2·8) by non-dimensional equivalents $\bar{\xi}$, $\bar{\eta}$, $\bar{W}$, $\bar{P}$, $\bar{\tau}$ and ζ. We write

$$z = h\zeta, \quad \beta h = a, \tag{2·13}$$

so that the liquid is now contained in the range $0 \leqslant \zeta \leqslant 1$. Also

$$\left.\begin{aligned} \xi = \Omega r_o \bar{\xi}, \quad \eta = \Omega r_o \bar{\eta}, \quad W = \tfrac{1}{2}\Omega a r_o \bar{W}, \\ P = \rho_0 a r_o h \Omega^2 \bar{P}, \quad \tau = \rho_0 a r_o \Omega^2 g^{-1} \alpha^{-1} \bar{\tau}. \end{aligned}\right\} \tag{2·14}$$

The following non-dimensional parameters are also introduced

$$\left.\begin{aligned} F = \tfrac{1}{2}\sigma'/\Omega, \quad f = \tfrac{1}{2}\sigma/\Omega, \quad R = \Omega\rho_0 h^2/\mu, \\ K = \Omega\rho_0 h^2 c_v/k, \quad R_H = g\alpha h\Theta_H(4\rho_0\Omega^2)^{-1}, \quad R_v = g\alpha\Theta_v(4\rho_0\Omega^2)^{-1}, \end{aligned}\right\} \tag{2·15}$$

so that

$$F = f + mR_H\zeta. \tag{2·16}$$

Here R is essentially a Reynolds number for the flow, K is a Péclet number; R_H is the parameter already introduced in (1·14), and f, F are non-dimensional frequencies.

The five equations (2·4) to (2·8) now become (with D written for the operator $d/d\zeta$)

$$F\bar{\xi} + (1+2R_H\zeta)\,\bar{\eta} = (2iR)^{-1}(D^2 - a^2)\,\bar{\xi}, \tag{2·17}$$

$$F\bar{\eta} + (1+2R_H\zeta)\,\bar{\xi} = -a^2 i\bar{P} + (2iR)^{-1}(D^2 - a^2)\,\bar{\eta}, \tag{2·18}$$

$$F\bar{W} + i\bar{\tau} = iD\bar{P} + (2iR)^{-1}(D^2 - a^2)\,\bar{W}, \tag{2·19}$$

$$\bar{\eta} + D\bar{W} = 0, \tag{2·20}$$

$$F\bar{\tau} - mR_H\bar{P} - iR_v\bar{W} = (2iK)^{-1}(D^2 - a^2)\,\bar{\tau}. \tag{2·21}$$

If $\bar{\xi}$, $\bar{\eta}$, $\bar{P}$, $\bar{\tau}$ are eliminated from these equations, a differential equation of eighth order is obtained for $\overline{W}$. The elimination is somewhat laborious, and the resulting equation is too complicated for easy interpretation. It is therefore desirable to make certain approximations based on the orders of magnitude of certain of the parameters. Typical experimental values are

$$R = 300, \quad f = -0{\cdot}05, \quad R_H = 0{\cdot}1, \quad K = 2000,$$

so that $F = -0{\cdot}05 + 0{\cdot}1m\zeta$. Accordingly $a^2/2R$ and $a^2/2K$ will be regarded as negligible compared with F; since $\mathrm{D}F$ is comparable with F/a, at the same time $(\mathrm{D}F)^2$, or $(mR_H)^2$, will be neglected compared with $2RF^3$. Again, complications in the eliminations are very appreciably reduced and no serious error is involved if in equations (2·17) and (2·18) the factor $1+2R_H\zeta$ is replaced by unity, and this will be done. It is not possible similarly to neglect the part $mR_H\zeta$ of F (equation (2·16)), since we shall be concerned with progressive waves travelling slowly in the same direction as the rotation. This implies a negative value of f, and therefore a possibility that F vanishes within the range $0 \leqslant \zeta \leqslant 1$, giving rise to singularities in the solutions of the equations.

When the approximations indicated are made, the differential equation for $\overline{W}$ is found in the form

$$-\frac{\mathrm{i}}{R\mathscr{P}}F\mathrm{D}^8\overline{W} + \frac{3\mathrm{i}mR_H}{R\mathscr{P}}\mathrm{D}^7\overline{W} - 2F^2\left(1+\frac{2}{\mathscr{P}}\right)\mathrm{D}^6\overline{W} + 4FmR_H\left(1-\frac{1}{\mathscr{P}}\right)\mathrm{D}^5\overline{W}$$
$$+4\mathrm{i}F\left\{2F^2 - \frac{1-F^2}{\mathscr{P}}\right\}R\mathrm{D}^4\overline{W} + 12\mathrm{i}mR_H(1+F^2)\frac{R}{\mathscr{P}}\mathrm{D}^3\overline{W} - 8F^2(1-F^2)R^2\mathrm{D}^2\overline{W}$$
$$+16FmR_HR^2\mathrm{D}\overline{W} + 8F^2R^2a^2(R_v - F^2)\,\overline{W} = 0, \tag{2·22}$$

where $\mathscr{P}$ denotes the Prandtl number K/R. In solving this equation it must be borne in mind that F, mR_H, and $\mathscr{P}^{-1}$ are small, and R^{-1} is very small.

3. Solution of the equation for $\overline{W}$

First consider solutions of equation (2·22) which can be expressed as a series of ascending powers of R^{-1}

$$\overline{W}(\zeta) = \overline{W}^{(0)}(\zeta) + R^{-1}\overline{W}^{(1)}(\zeta) + \dots. \tag{3·1}$$

If (3·1) is substituted in equation (2·22), a comparison of corresponding powers of R^{-1} leads to the equations

$$(1-F^2)\,\mathrm{D}^2\overline{W}^{(0)} - \frac{2}{F}mR_H\mathrm{D}\overline{W}^{(0)} + a^2(F^2 - R_v)\,\overline{W}^{(0)} = 0, \tag{3·2}$$

$$4\mathrm{i}F\left(2F^2 - \frac{1-F^2}{\mathscr{P}}\right)\mathrm{D}^4\overline{W}^{(0)} + 12\mathrm{i}mR_H(1+F^2)\frac{1}{\mathscr{P}}\mathrm{D}^3\overline{W}^{(0)} - 8F^2(1-F^2)\,\mathrm{D}^2\overline{W}^{(1)}$$
$$+16FmR_H\mathrm{D}\overline{W}^{(1)} + 8Fa^2(R_v - F^2)\,\overline{W}^{(1)} = 0, \quad \text{etc.} \tag{3·3}$$

The initial equation has two solutions, so this method provides just two solutions of equation (2·22). In solving equation (3·2) we may approximate by neglecting F^2, which is small compared with unity, in the first and third terms. This replacing of $(1-F^2)$ by unity means that two singularities of equation (3·2) are ignored. These two singularities are given by

$$\zeta_3 = \frac{1-f}{mR_H}, \quad \zeta_4 = \frac{-1-f}{mR_H}. \tag{3·4}$$

Since f is, in the most important case, small negative quantity of order $-0{\cdot}05$, the singularity ξ_4 cannot fall within the range $0 \leqslant \zeta \leqslant 1$, and the singularity ζ_3 will fall within this range only if m and R_H are large. The critical values of R_H for given m which are obtained later are such that in all cases the product of m and R_H is substantially less than unity, so that the neglect of the singularity ζ_3 is justified. The two singularities $\zeta_1 = \zeta_2 = -f/mR_H$ (corresponding to $F = 0$), which will lie within the range $0 \leqslant \zeta \leqslant 1$ if f is negative and $|f| < mR_H$, are retained. The modified form of equation (3·2) to be solved is then

$$\mathrm{D}^2\overline{W}^{(0)} - \frac{2}{F} mR_H \mathrm{D}\overline{W}^{(0)} - a^2R_v\overline{W}^{(0)} = 0, \tag{3·5}$$

where $F = f + mR_H\zeta$.

Equation (3·5) is precisely that solved by Davies (1956). His method was to change the variable so that the origin is transformed to the singular point, leading to the equation

$$\frac{\mathrm{d}^2\overline{W}^{(0)}}{\mathrm{d}\zeta^{*2}} - \frac{2}{\zeta^*}\frac{\mathrm{d}\overline{W}^{(0)}}{\mathrm{d}\zeta^*} - a^2R_v\overline{W}^{(0)} = 0, \tag{3·6}$$

where
$$\zeta^* = \zeta + \frac{f}{mR_H} = \zeta - \zeta_1. \tag{3·7}$$

The complete solution of (3·6) was shown to be

$$\overline{W}^{(0)} = A(\sin\psi - \psi\cos\psi) + B(\psi\sin\psi + \cos\psi), \tag{3·8}$$

where $\psi = \mathrm{i}\zeta^* aR_v^{\frac{1}{2}}$.

Solutions for $\overline{W}^{(1)}$ have not been found, since the large value of R, usually around 300, implies that $R^{-1}\overline{W}^{(1)} \ll \overline{W}^{(0)}$.

In order to obtain six other independent integrals of equation (2·22) we next make a transformation

$$\overline{W} = W_0(\zeta)\exp\{R^{\frac{1}{2}}Q(\zeta)\}. \tag{3·9}$$

The derivatives of various orders of $\overline{W}$ then become

$$\left.\begin{aligned} \mathrm{D}\overline{W} &= \{W_0(\zeta)\,R^{\frac{1}{2}}Q'(\zeta) + W_0'(\zeta)\}\,\mathrm{e}^{R^{\frac{1}{2}}Q(\zeta)}, \\ \mathrm{D}^2\overline{W} &= \{W_0(RQ'^2 + R^{\frac{1}{2}}Q'') + 2W_0'R^{\frac{1}{2}}Q' + W_0''\}\mathrm{e}^{R^{\frac{1}{2}}Q}, \\ \mathrm{D}^n\overline{W} &= [W_0\{R^{\frac{1}{2}n}Q'^n + (1+2+\ldots+n-1)\,R^{\frac{1}{2}(n-1)}Q'^{n-2}Q'' + \ldots\} \\ &\quad + W_0'\{nR^{\frac{1}{2}(n-1)}Q'^{n-1} + \ldots\} + \ldots]\,\mathrm{e}^{R^{\frac{1}{2}}Q}, \end{aligned}\right\} \tag{3·10}$$

where the prime now denotes differentiation with respect to ζ. Substituting (3·9) and (3·10) in (2·22) and equating to zero the coefficient of the highest power of R (in this case R^3) we find the equation

$$\left(\frac{\mathrm{i}}{\mathscr{P}}Q'^2 + 2F\right)\{(Q'^2 - 2\mathrm{i}F)^2 + 4\}\,Q'^2 = 0. \tag{3·11}$$

The solutions corresponding to $Q'^2 = 0$ are those already obtained in equation (3·8). Neglecting these, the values of Q' are given by

$$\left.\begin{aligned} Q_1' &= (1+\mathrm{i})\,(1+F)^{\frac{1}{2}},\\ Q_2' &= (1-\mathrm{i})\,(1-F)^{\frac{1}{2}},\\ Q_3' &= -(1+\mathrm{i})\,(1+F)^{\frac{1}{2}},\\ Q_4' &= -(1-\mathrm{i})\,(1-F)^{\frac{1}{2}},\\ Q_5' &= (1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}},\\ Q_6' &= -(1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}. \end{aligned}\right\} \tag{3·12}$$

We next have to determine the corresponding solutions for the function $\overline{W}_0(\zeta)$ introduced in equation (3·9). This is achieved by equating to zero the coefficient of the next highest power of R in equation (2·22), i.e. that of $R^{\frac{5}{3}}$.

The resulting equation is

$$\begin{aligned} &-\frac{\mathrm{i}F}{\mathscr{P}}\{28(Q')^6\,Q''\overline{W}_0+8(Q')^7\overline{W}_0'\}+\frac{3\mathrm{i}mR_H}{\mathscr{P}}(Q')^7\overline{W}_0-30F^2\left(1+\frac{2}{\mathscr{P}}\right)(Q')^4Q''\overline{W}_0\\ &-12F^2\left(1+\frac{2}{\mathscr{P}}\right)(Q')^5\overline{W}_0'+4FmR_H\left(1-\frac{1}{\mathscr{P}}\right)(Q')^5\overline{W}_0\\ &+\frac{4\mathrm{i}F}{\mathscr{P}}(2F^2\mathscr{P}-1+F^2)\,\{6(Q')^2\,Q''\overline{W}_0+4(Q')^3\,\overline{W}_0'\}+\frac{12\mathrm{i}mR_H}{\mathscr{P}}(1+F^2)\,(Q')^3\overline{W}_0\\ &-8F^2(1-F^2)\,\{Q''\overline{W}_0+2Q'\overline{W}_0'\}+16FmR_H\,Q'\overline{W}_0=0, \end{aligned} \tag{3·13}$$

which, on replacing $(Q_i')^2$ by $2\mathrm{i}(1+F)$ for $i=1,3,$

by $-2\mathrm{i}(1-F)$ for $i=2,4$

and by $2\mathrm{i}F\mathscr{P}$ for $i=5,6$

leads to the equation

$$\begin{aligned} \frac{\overline{W}_{0i}'}{\overline{W}_{0i}} = -\frac{Q_i''}{Q_i'}&\left[\frac{-224\frac{F}{\mathscr{P}}(1+F)^3+120F^2\left(1+\frac{2}{\mathscr{P}}\right)(1+F)^2-48\frac{F}{\mathscr{P}}(2F^2\mathscr{P}-1+F^2)\,(1+F)-8F^2(1-F^2)}{-64\frac{F}{\mathscr{P}}(1+F)^3+48F^2(1+F)^2-32\frac{F}{\mathscr{P}}(2F^2\mathscr{P}-1+F^2)\,(1+F)-16F^2(1-F^2)}\right]\\ &-mR_H\left[\frac{\frac{24}{\mathscr{P}}(1+F)^3-16F\left(1-\frac{1}{\mathscr{P}}\right)(1+F)^2-\frac{24}{\mathscr{P}}(1+F)\,(1+F^2)+16F}{-64\frac{F}{\mathscr{P}}(1+F)^3+48F^2(1+F)^2-32\frac{F}{\mathscr{P}}(2F^2\mathscr{P}-1+F^2)\,(1+F)-16F^2\,(1-F^2)}\right] \end{aligned}$$

for $i=1,3.$ (3·14)

On retaining only the terms of order unity, together with $F\mathscr{P}$, this simplifies to

$$\frac{\overline{W}_{0i}'}{\overline{W}_{0i}} = -\frac{mR_H}{2(1+F)}\left[\frac{-176+112F\mathscr{P}}{-32+32F\mathscr{P}}\right]-mR_H\left[\frac{64-32F\mathscr{P}}{-32+32F\mathscr{P}}\right]=\frac{-3mR_H}{4(1+F)},$$

giving

$$\overline{W}_{0i} = (1+F)^{-\frac{3}{4}} \quad \text{for} \quad i=1,3. \tag{3·15}$$

By use of exactly similar approximations the other solutions for $\overline{W}_{0i}$ are found to be

$$\overline{W}_{0i} = (-1+F)^{-\frac{3}{4}} \quad \text{for} \quad i = 2, 4, \tag{3·16}$$

$$\overline{W}_{0i} = F^{-\frac{3}{4}}\{F^2(\mathscr{P}-1)^2-1\}^{-1} \quad \text{for} \quad i = 5, 6. \tag{3·17}$$

The eighth independent solutions of equation (2·22) obtained by asymptotic series are therefore

$$\left.\begin{aligned}
\overline{W}_1 &= (1+F)^{-\frac{3}{4}} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}]\,\mathrm{d}\zeta\right\},\\
\overline{W}_2 &= (-1+F)^{-\frac{3}{4}} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}]\,\mathrm{d}\xi\right\},\\
\overline{W}_3 &= (1+F)^{-\frac{3}{4}} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[-(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}]\,\mathrm{d}\zeta\right\},\\
\overline{W}_4 &= (-1+F)^{-\frac{3}{4}} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[-(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}]\,\mathrm{d}\zeta\right\},\\
\overline{W}_5 &= F^{-\frac{3}{4}}\{F^2(\mathscr{P}-1)^2-1\}^{-1} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[(1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}]\,\mathrm{d}\zeta\right\},\\
\overline{W}_6 &= F^{-\frac{3}{4}}\{F^2(\mathscr{P}-1)^2-1\}^{-1} \exp\left\{\int_{\zeta_1}^{\zeta} R^{\frac{1}{2}}[-(1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}]\,\mathrm{d}\zeta\right\},\\
&\left.\begin{aligned}\overline{W}_7 &= \sin\psi-\psi\cos\psi\\ \overline{W}_8 &= \psi\sin\psi+\cos\psi\end{aligned}\right\}, \quad \text{where} \quad \psi = \mathrm{i}(\zeta-\zeta_1)\,aR_v^{\frac{1}{2}},
\end{aligned}\right\} \tag{3·18}$$

and the complete solution for $\overline{W}$ is

$$\overline{W} = \sum_1^8 c_i \overline{W}_i, \tag{3·19}$$

where the c_i are constants.

It is evident that at least two of the solutions, namely $\overline{W}_5$ and $\overline{W}_6$, have a branch point at $\zeta = \zeta_1$, i.e. the point at which F vanishes. If this point lies between $\zeta = 0$ and $\zeta = 1$, the choice of the correct branch is not at once clear. Similar difficulties arising in the solutions of the Orr–Sommerfeld equation for parallel flows in hydrodynamics have been discussed by Lin (1955). For the present, however, we merely state the requirement that one must choose a path of integration from $\zeta = 0$ to $\zeta = 1$ in the complex ζ-plane such that the real part of Q increases monotonically. An indication of a suitable choice of path is given in appendix A.

4. The boundary conditions

The boundary conditions appropriate to the problem have been set out above in the form

$$\xi = \eta = W = 0 \quad \text{at} \quad z = 0, \tag{2·10}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = \frac{\mathrm{d}\eta}{\mathrm{d}z} = W = 0 \quad \text{at} \quad z = h, \tag{2·11}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = 0 \quad \text{at} \quad z = 0 \quad \text{and at} \quad z = h. \tag{2·12}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The task of converting these conditions to expressions containing W only is rather formidable, and it is found more convenient to apply the conditions directly as they stand, assuming solutions for $\bar{\xi}$, $\bar{\eta}$ and $\bar{\tau}$ of the form

$$\left.\begin{aligned}\bar{\xi}_i &= \bar{\xi}_{0i}\exp\{R^{\frac{1}{2}}Q_i(\zeta)\},\\ \bar{\eta}_i &= \bar{\eta}_{0i}\exp\{R^{\frac{1}{2}}Q_i(\zeta)\},\\ \bar{\tau}_i &= \bar{\tau}_{0i}\exp\{R^{\frac{1}{2}}Q_i(\zeta)\},\end{aligned}\right\} \tag{4·1}$$

where $Q_i(\zeta)$, $i = 1, \ldots, 6$ are given by equation (3·12).

The functions $\bar{\xi}_{0i}$, $\bar{\eta}_{0i}$, $\bar{\tau}_{0i}$ are found in terms of $\bar{W}_{0i}$ by using equations (2·17) to (2·21), and the complete solutions for $\bar{\xi}$, $\bar{\eta}$, $\bar{\tau}$ are

$$\left.\begin{aligned}\bar{\xi} &= \Sigma c_i\bar{\xi}_i,\\ \bar{\eta} &= \Sigma c_i\bar{\eta}_i,\\ \bar{\tau} &= \Sigma c_i\bar{\tau}_i,\end{aligned}\right\} \tag{4·2}$$

where the c_i are identical with those of equation (3·19).

The functions $\bar{\xi}_{0i}$, $\bar{\eta}_{0i}$, $\bar{\tau}_{0i}$ are found to be, to the first approximation in powers of $R^{\frac{1}{2}}$,

$$\begin{aligned}
\bar{\xi}_{01} &= -R^{\frac{1}{2}}\{(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\}\bar{W}_{01}, & \bar{\xi}_{02} &= R^{\frac{1}{2}}\{(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}\}\bar{W}_{02},\\
\bar{\xi}_{03} &= -R^{\frac{1}{2}}\{-(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\}\bar{W}_{03}, & \bar{\xi}_{04} &= R^{\frac{1}{2}}\{-(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}\}\bar{W}_{04},\\
\bar{\xi}_{05} &= -\frac{R^{\frac{1}{2}}}{\mathscr{P}-1}\{(1+\mathrm{i})\,F^{-\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}\}\bar{W}_{05}, & \bar{\xi}_{06} &= -\frac{R^{\frac{1}{2}}}{\mathscr{P}-1}\{-(1+\mathrm{i})\,F^{-\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}\}\bar{W}_{06};\\
\bar{\eta}_{01} &= -R^{\frac{1}{2}}\{(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\}\bar{W}_{01}, & \bar{\eta}_{02} &= -R^{\frac{1}{2}}\{(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}\}\bar{W}_{02},\\
\bar{\eta}_{03} &= -R^{\frac{1}{2}}\{-(1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\}\bar{W}_{03}, & \bar{\eta}_{04} &= -R^{\frac{1}{2}}\{-(1-\mathrm{i})\,(1-F)^{\frac{1}{2}}\}\bar{W}_{04},\\
\bar{\eta}_{05} &= -R^{\frac{1}{2}}\{(1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}\}\bar{W}_{05}, & \bar{\eta}_{06} &= -R^{\frac{1}{2}}\{-(1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}\}\bar{W}_{06};\\
\bar{\tau}_{01} &= \{F(\mathscr{P}-1)-1\}^{-1}\,\mathrm{i}\mathscr{P}R_v\bar{W}_{01}, & \bar{\tau}_{02} &= \{F(\mathscr{P}-1)+1\}^{-1}\,\mathrm{i}\mathscr{P}R_v\bar{W}_{02},\\
\bar{\tau}_{03} &= \{F(\mathscr{P}-1)-1\}^{-1}\,\mathrm{i}\mathscr{P}R_v\bar{W}_{03}, & \bar{\tau}_{04} &= \{F(\mathscr{P}-1)+1\}^{-1}\,\mathrm{i}\mathscr{P}R_v\bar{W}_{04},\\
\bar{\tau}_{05} &= -2\mathrm{i}R\frac{\mathscr{P}}{\mathscr{P}-1}\{F^2(\mathscr{P}-1)^2-1\}\bar{W}_{05}, & \bar{\tau}_{06} &= -2\mathrm{i}R\frac{\mathscr{P}}{\mathscr{P}-1}\{F^2(\mathscr{P}-1)^2-1\}\bar{W}_{06}.
\end{aligned} \tag{4·3}$$

The application of the boundary conditions (2·10) to (2·12) above leads to a consistency equation for the non-vanishing of the coefficients c_i, and this equation takes the form

$$\begin{vmatrix}
\bar{W}_1(0) & \bar{W}_2(0) & \bar{W}_3(0) & \bar{W}_4(0) & \bar{W}_5(0) & \bar{W}_6(0) & \bar{W}_7(0) & \bar{W}_8(0)\\
\bar{W}_1(1) & \bar{W}_2(1) & & & & & & \bar{W}_8(1)\\
\bar{\eta}_1(0) & \bar{\eta}_2(0) & & & & & & \bar{\eta}_8(0)\\
\bar{\xi}_1(0) & \bar{\xi}_2(0) & & & & & & \bar{\xi}_8(0)\\
\mathrm{D}\bar{\eta}_1(1) & \mathrm{D}\bar{\eta}_2(1) & & & & & & \mathrm{D}\bar{\eta}_8(1)\\
\mathrm{D}\bar{\xi}_1(1) & \mathrm{D}\bar{\xi}_2(1) & & & & & & \mathrm{D}\bar{\xi}_8(1)\\
\mathrm{D}\bar{\tau}_1(1) & \mathrm{D}\bar{\tau}_2(1) & & & & & & \mathrm{D}\bar{\tau}_8(1)\\
\mathrm{D}\bar{\tau}_1(0) & \mathrm{D}\bar{\tau}_2(0) & & & & & & \mathrm{D}\bar{\tau}_8(0)
\end{vmatrix} = 0. \tag{4·4}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

All the terms in the first column have a factor $\exp\left\{R^{\frac{1}{2}}\int_{\zeta_1}^{\zeta} Q_1'\,\mathrm{d}\zeta\right\}$, so those terms evaluated at $\zeta = 0$ have a factor

$$\exp\left\{R^{\frac{1}{2}}\int_{\zeta_1}^{0} (1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\,\mathrm{d}\zeta\right\},$$

and those terms evaluated at $\zeta = 1$ have a factor

$$\exp\left\{R^{\frac{1}{2}}\int_{\zeta_1}^{1} (1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\,\mathrm{d}\zeta\right\}.$$

This feature enables us to simplify the determinant considerably, for if the first column be divided by

$$\exp\left\{R^{\frac{1}{2}}\int_{\zeta_1}^{1} (1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\,\mathrm{d}\zeta\right\},$$

the terms evaluated at $\zeta = 0$ have a factor

$$\exp\left\{-R^{\frac{1}{2}}\int_{0}^{1} (1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\,\mathrm{d}\zeta\right\},$$

and, provided the path of integration chosen is such that

$$\int_{0}^{1} (1+\mathrm{i})\,(1+F)^{\frac{1}{2}}\,\mathrm{d}\zeta > 0,$$

this factor means that these terms may be neglected by comparison with the terms evaluated at $\zeta = 1$. Since F is much smaller than unity the real axis provides such a path in this case. Similar simplifications may be made in columns 2, 3 and 4, the choice of path of integration causing no difficulty, but when the method is applied to columns 5 and 6 a factor

$$\exp\left\{-R^{\frac{1}{2}}\int_{0}^{1} (1+\mathrm{i})\,F^{\frac{1}{2}}\mathscr{P}^{\frac{1}{2}}\,\mathrm{d}\zeta\right\}$$

arises. In order to justify the neglect of such terms by comparison with unity it is required that the path of integration chosen should be such that $\int F^{\frac{1}{2}}\,\mathrm{d}\xi$ taken along the path should increase monotonically from $\zeta = 0$ to $\zeta = 1$, the condition referred to above.

As a consequence of these simplifications, equation (4·4) becomes

$$\begin{vmatrix}
0 & 0 & \overline{W}_{03}(0) & \overline{W}_{04}(0) & 0 & \overline{W}_{06}(0) & \overline{W}_7(0) & \overline{W}_8(0) \\
\overline{W}_{01}(1) & \overline{W}_{02}(1) & 0 & 0 & \overline{W}_{05}(1) & 0 & \overline{W}_7(1) & \overline{W}_8(1) \\
0 & 0 & \bar\eta_{03}(0) & \bar\eta_{04}(0) & 0 & \bar\eta_{06}(0) & \bar\eta_7(0) & \bar\eta_8(0) \\
0 & 0 & \bar\xi_{03}(0) & \bar\xi_{04}(0) & 0 & \bar\xi_{06}(0) & \bar\xi_7(0) & \bar\xi_8(0) \\
R^{\frac{1}{2}}Q_1'\bar\eta_{01}(1) & R^{\frac{1}{2}}Q_2'\bar\eta_{02}(1) & 0 & 0 & R^{\frac{1}{2}}Q_5'\bar\eta_{05}(1) & 0 & \mathrm{D}\bar\eta_7(1) & \mathrm{D}\bar\eta_8(1) \\
R^{\frac{1}{2}}Q_1'\bar\xi_{01}(1) & R^{\frac{1}{2}}Q_2'\bar\xi_{02}(1) & 0 & 0 & R^{\frac{1}{2}}Q_5'\bar\xi_{05}(1) & 0 & \mathrm{D}\bar\xi_7(1) & \mathrm{D}\bar\xi_8(1) \\
R^{\frac{1}{2}}Q_1'\bar\tau_{01}(1) & R^{\frac{1}{2}}Q_2'\bar\tau_{02}(1) & 0 & 0 & R^{\frac{1}{2}}Q_5'\bar\tau_{05}(1) & 0 & \mathrm{D}\bar\tau_7(1) & \mathrm{D}\bar\tau_8(1) \\
0 & 0 & R^{\frac{1}{2}}Q_3'\bar\tau_{03}(0) & R^{\frac{1}{2}}Q_4'\bar\tau_{04}(0) & 0 & R^{\frac{1}{2}}Q_6\bar\tau_{06}(0) & \mathrm{D}\bar\tau_7(0) & \mathrm{D}\bar\tau_8(0)
\end{vmatrix} = 0, \quad (4{\cdot}5)$$

where the derivatives $\mathrm{D}\bar\eta_i$, $\mathrm{D}\bar\xi_i$, $\mathrm{D}\bar\tau_i$ have been replaced by the leading term in their expansions in powers of $R^{\frac{1}{2}}$.

Evaluation of the determinant is now quite straightforward, and the resulting form of equation (4·5) is

$$[(1+\psi_0\psi_1)\sin(\psi_0-\psi_1)-(\psi_0-\psi_1)\cos(\psi_0-\psi_1)] \\ -\frac{R^{-\frac{1}{2}}}{4f}\left[(1-\mathrm{i})\frac{1-f}{(1+f)^{\frac{1}{2}}}-(1+\mathrm{i})\frac{1+f}{(1-f)^{\frac{1}{2}}}\right][(\mathrm{D}\psi_0)\{\psi_0\sin(\psi_0-\psi_1)+\psi_0\psi_1\cos(\psi_0-\psi_1)\}] \\ +O(R^{-1})=0, \quad (4\cdot6)$$

where $\psi_0=-\mathrm{i}\zeta_1 aR_v^{\frac{1}{2}}, \quad \psi_1=\mathrm{i}(1-\zeta_1)\,aR_v^{\frac{1}{2}}.$

The vanishing of the term independent of R constitutes the condition established by Davies (1956).

Before proceeding further, an assumption made in the evaluation of the determinant must be stated explicitly. In certain of the terms (see (3·18), (4·3)) a factor $\{F^2(\mathscr{P}-1)^2-1\}^{-1}$ or $\{F(\mathscr{P}-1)-1\}^{-1}$ occurs, and these factors have been assumed small by comparison with $R^{\frac{1}{2}}$. However, since $F=f+mR_H\zeta$, where f is usually a small negative quantity, it seems that F will attain the value $\dfrac{1}{(\mathscr{P}-1)}$ within the range $0\leqslant\zeta\leqslant 1$ for certain values of m and R_H, suggesting the presence within the liquid of a layer where velocities and temperature gradients are large. Unfortunately, experimental data for liquids of differing Prandtl number are at present insufficient to permit a useful examination for such a phenomenon. A discussion of the mathematical implications is given in appendix B.

5. Derivation of a solution for f and comparison with experiment

Equation (4·6) provides us with one relationship between the parameters f, β, R_v, R_H and R. A second is obtained by considering the consequences of satisfying the boundary condition $u=0$ at $r=r_0$. The appropriate expression for u is derived from equations (2·1a) and (2·17), namely

$$2u=\left\{\frac{mJ_m(\beta r)}{\beta r}+\left(F+\frac{a^2}{2\mathrm{i}R}\right)J'_m(\beta r)\right\}\xi(\zeta)-\frac{1}{2\mathrm{i}R}J'_m(\beta r)\frac{\mathrm{d}^2\xi}{\mathrm{d}\zeta^2}.$$

Now u must vanish at $r=r_o$ for all ζ, and the second derivative $\mathrm{d}^2\xi/\mathrm{d}\zeta^2$ is of order R only in the neighbourhood of $\zeta=0$ and $\zeta=1$, so the approximate condition we shall use is

$$\frac{mJ_m(\beta r_o)}{\beta r_o}+fJ'_m(\beta r_o)=0 \quad (5\cdot1)$$

i.e. F is replaced by the quantity f, which is independent of ζ. Writing equation (4·6) in a more explicit form, we have

$$\left\{\mathrm{i}aR_v^{\frac{1}{2}}+R^{-\frac{1}{2}}\mathrm{i}a^3R_v^{\frac{3}{2}}\frac{1}{4mR_H}\left(1+\frac{f}{mR_H}\right)\left[(1-\mathrm{i})\frac{1-f}{(1+f)^{\frac{1}{2}}}-(1+\mathrm{i})\frac{1+f}{(1-f)^{\frac{1}{2}}}\right]\right\}\cos(\mathrm{i}aR_v^{\frac{1}{2}}) \\ -\left\{1-\frac{fa^2R_v}{mR_H}\left(1+\frac{f}{mR_H}\right)+R^{-\frac{1}{2}}a^2R_v\frac{1}{4mR_H}\left[(1-\mathrm{i})\frac{1-f}{(1+f)^{\frac{1}{2}}}-(1+\mathrm{i})\frac{1+f}{(1-f)^{\frac{1}{2}}}\right]\right\}\sin(\mathrm{i}aR_v^{\frac{1}{2}})=0. \quad (5\cdot2)$$

We now seek a solution of equations (5·1) and (5·2) in which βr_o and f are each expressed in terms of a series in ascending powers of $R^{-\frac{1}{2}}$, putting

$$\left.\begin{aligned}\beta r_o&=\beta_1+R^{-\frac{1}{2}}\beta_2+\dots,\\ f&=F_1+R^{-\frac{1}{2}}F_2+\dots.\end{aligned}\right\} \quad (5\cdot3)$$

From the Taylor expansion for the Bessel function $J_m(x)$ it follows at once that

$$J_m(\beta_1+R^{-\frac{1}{2}}\beta_2+\ldots) = J_m(\beta_1)+R^{-\frac{1}{2}}\beta_2 J'_m(\beta_1)+O(R^{-1}) \tag{5·4}$$

and

$$J'_m(\beta_1+R^{-\frac{1}{2}}\beta_2+\ldots) = J'_m(\beta_1)-R^{-\frac{1}{2}}\beta_2\left\{\frac{1}{\beta_1}J'_m(\beta_1)+\left(1-\frac{m^2}{\beta_1^2}\right)J_m(\beta_1)\right\},$$

so equation (5·1) becomes, retaining only terms of order unity and $R^{-\frac{1}{2}}$,

$$m\{J_m(\beta_1)+R^{-\frac{1}{2}}\beta_2 J'_m(\beta_1)\} + (F_1+R^{-\frac{1}{2}}F_2)(\beta_1+R^{-\frac{1}{2}}\beta_2)\left[J'_m(\beta_1)-R^{-\frac{1}{2}}\beta_2\left\{\frac{1}{\beta_1}J'_m(\beta_1)+\left(1-\frac{m^2}{\beta_1^2}\right)J_m(\beta_1)\right\}\right] = 0. \tag{5·5}$$

By equating to zero the terms of order unity and order $R^{-\frac{1}{2}}$ respectively we obtain the equations

$$mJ_m(\beta_1)+F_1\beta_1 J'_m(\beta_1) = 0, \tag{5·6}$$

$$\beta_2[mJ'_m(\beta_1)+(F_1/\beta_1)(m^2-\beta_1^2)J_m(\beta_1)]+F_2\beta_1 J'_m(\beta_1) = 0. \tag{5·7}$$

Two further equations are obtained by substituting the expressions (5·3) in (5·2), using the relation $a = \beta h = (h/r_o)(\beta_1+R^{-\frac{1}{2}}\beta_2+\ldots)$ and equating to zero the terms of order unity and $R^{-\frac{1}{2}}$ respectively,

$$\beta_1 x_1\cos\beta_1 x_1-\left\{1+\frac{\beta_1^2 x_1^2 F_1}{mR_H}\left(1+\frac{F_1}{mR_H}\right)\right\}\sin\beta_1 x_1 = 0, \tag{5·8}$$

$$\beta_2\left[\left\{1-\frac{2F_1}{mR_H}\left(1+\frac{F_1}{mR_H}\right)\right\}\sin\beta_1 x_1-\frac{F_1}{mR_H}\left(1+\frac{F_1}{mR_H}\right)\beta_1 x_1\cos\beta_1 x_1\right]$$
$$-F_2\frac{\beta_1}{mR_H}\left(1+\frac{2F_1}{mR_H}\right)\sin\beta_1 x_1-\frac{\beta_1}{4mR_H}\left\{(1-\mathrm{i})\frac{1-F_1}{(1+F_1)^{\frac{1}{2}}}-(1-\mathrm{i})\frac{1+F_1}{(1-F_1)^{\frac{1}{2}}}\right\}$$
$$\times\left\{\sin\beta_1 x_1-\left(1+\frac{F_1}{mR_H}\right)\beta_1 x_1\cos\beta_1 x_1\right\} = 0, \tag{5·9}$$

where $\mathrm{i}hR^{\frac{1}{2}}/r_o$ is written as x_1. We now have a set of four equations (5·6) to (5·9) connecting the four unknowns F_1, F_2, β_1, β_2. F_1 and β_1 are found by solving equations (5·6) and (5·8); this has been done approximately by Davies (1956). These results are then used in equations (5·7) and (5·9) in order to find a solution for F_2 in terms of the other parameters, m, R_H, R_v, K and R, of the problem.

Davies's solutions for F_1 and β_1 are

$$\frac{F_1}{mR_H} = -0·5-0·0229\frac{h}{r_o}R_v^{\frac{1}{2}}R_H x_{sm}\pm 0·214\left\{\frac{(1+\frac{1}{2}R_H)\,hR_v^{\frac{1}{2}}x_{sm}}{r_o}-2·4\right\}^{\frac{1}{2}}, \tag{5·10}$$

$$\beta_1 = x_{sm}\left(1-\frac{F_1}{m}\right), \tag{5·11}$$

where x_{sm} is the sth zero of $J_m(x) = 0$.

The second term on the right-hand side of equation (5·10) is much smaller than 0·5 and is therefore neglected. The third term is imaginary for $\{(1+\frac{1}{2}R_H)\,hR_v^{\frac{1}{2}}x_{sm}\}/r_o < 2·4$, and since complex values of f imply instability it follows that the vanishing of this third term provides a stability criterion in the first approximation. Writing $F_1 = F_{1\mathscr{R}}+\mathrm{i}F_{1\mathscr{I}}$, where $F_{1\mathscr{R}}$ and $F_{1\mathscr{I}}$ are the real and imaginary parts respectively of F_1, we have $F_{1\mathscr{I}}$ equal to zero for $\{(1+\frac{1}{2}R_H)\,hR_v^{\frac{1}{2}}x_{sm}\}/r_o > 2·4$ and $F_{1\mathscr{R}}$ equal to 0·5 for $F_{1\mathscr{I}}$ non-zero.

The stability criterion is the vanishing of the imaginary part of f, i.e.

$$F_{1\mathscr{I}}+R^{-\frac{1}{2}}F_{2\mathscr{I}}=0, \tag{5·12}$$

where $F_{2\mathscr{I}}$ is the imaginary part of F_2, so the next requirement is a solution for F_2, the second term in the series (5·3). An explicit equation for F_2 arises when β_2 is eliminated between (5·7) and (5·9), and is

$$F_2=\frac{G(\beta_1,F_1)}{H(\beta_1,F_1)},$$

where

$$G\equiv\frac{\beta_1}{4mR_H}\left\{(1-\mathrm{i})\frac{1-F_1}{(1+F_1)^{\frac{1}{2}}}-(1+\mathrm{i})\frac{1+F_1}{(1-F_1)^{\frac{1}{2}}}\right\}\left\{\sin\beta_1x_1-\left(1+\frac{F_1}{mR_H}\right)\beta_1x_1\cos\beta_1x_1\right\}$$
$$\times\left\{mJ'_m(\beta_1)+\frac{F_1}{\beta_1}(m^2-\beta_1^2)J_m(\beta_1)\right\},$$
$$H\equiv\left[\left\{1-\frac{2F_1}{mR_H}\left(1+\frac{F_1}{mR_H}\right)\right\}\sin\beta_1x_1-\frac{F_1}{mR_H}\left(1+\frac{F_1}{mR_H}\right)\beta_1x_1\cos\beta_1x_1\right]\beta_1J'_m(\beta_1)$$
$$+\frac{\beta_1}{mR_H}\left(1+\frac{2F_1}{mR_H}\right)\sin\beta_1x_1\left\{mJ'_m(\beta_1)+\frac{F_1}{\beta_1}(m^2-\beta_1^2)J_m(\beta_1)\right\}. \tag{5·13}$$

Now if the value of $F_{1\mathscr{I}}$ from equation (5·10) is substituted in equation (5·12) an equation is found which is true when the imaginary part of f vanishes, so the values of the parameters satisfying this equation might be expected to have some critical value in the change-over from wave to spiral flow. This equation is

$$\frac{(1+\frac{1}{2}R_H)\,hR_v^{\frac{1}{2}}x_{sm}}{\mathrm{r}_0}=2\cdot4-\frac{(F_{2\mathscr{I}})^2}{(0\cdot214mR_H)^2R}. \tag{5·14}$$

The complicated and unsymmetrical nature of the solution (5·13) for F_2 makes this equation rather difficult to handle in its exact form, but certain approximations based on the assumption that $R_H\ll 1$ may be made, namely

(i) the factor $$\left\{(1-\mathrm{i})\frac{1-F_1}{(1+F_1)^{\frac{1}{2}}}-(1+\mathrm{i})\frac{1+F_1}{(1-F_1)^{\frac{1}{2}}}\right\}$$
is replaced by $-2\mathrm{i}$;

(ii) the first of the two terms comprising $H(\beta_1,F_1)$, being of order $R_H\times$ the second term, is neglected.

Thus an approximate solution for F_2 is given by

$$F_2=-2\mathrm{i}\left\{1-\left(1+\frac{F_1}{mR_H}\right)\beta_1x_1\cot\beta_1x_1\right\}\Big/4\left(1+\frac{2F_1}{mR_H}\right). \tag{5·15}$$

The imaginary part, $F_{2\mathscr{I}}$, may be written in the form

$$F_{2\mathscr{I}}=\frac{\frac{1}{4}(0\cdot856)^2\{(1+\frac{1}{2}R_H)\,y-2\cdot4\}\,y\coth y-0\cdot0916yR_H\{1-(\frac{1}{2}+0\cdot0229yR_H)\,y\coth y\}}{(0\cdot856)^2\{(1+\frac{1}{2}R_H)\,y-2\cdot4\}+(0\cdot0916yR_H)^2}, \tag{5·16}$$

where $$y=\frac{hR_v^{\frac{1}{2}}}{r_o}x_{sm}, \tag{5·17}$$

from which it appears that, except when $\{(1+\frac{1}{2}R_H)\,y-2\cdot4\}$ is very small, i.e. when $F_{1\mathscr{I}}$ is very small, a good approximation for $F_{2\mathscr{I}}$ is

$$F_{2\mathscr{I}}=\tfrac{1}{4}y\coth y. \tag{5·18}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Equation (5·14) then becomes

$$(1+\tfrac{1}{2}R_H)\,y = 2{\cdot}4-\frac{y^2\coth^2 y}{16(0{\cdot}214mR_H)^2 R} \tag{5·19}$$

and gives a criterion for stability in terms of R_v and R_H for a given wave-number. However, almost all experiments have suggested a critical dependence only on parameters identifiable with R_H, and so some sort of relation between R_v and R_H might be expected to exist. Lorenz (1953) and Davies (1956) have shown that there is in fact such a relation, and Davies derived an explicit form for it. His method was to expand the non-dimensionalized dependent variables in ascending powers of R_H, namely

$$\left.\begin{aligned} u &= R_H u_1 + R_H^2 u_2 + \dots,\\ v &= r + R_H v_1 + R_H^2 v_2 + \dots,\\ w &= R_H w_1 + R_H^2 w_2 + \dots,\\ \rho &= 1 + R_H \rho_1 + R_H^2 \rho_2 + \dots,\\ p &= p_0 + R_H p_1 + R_H^2 p_2 + \dots,\\ T &= \tau_0 + R_H \tau_1 + R_H^2 \tau_2 + \dots, \end{aligned}\right\} \tag{5·20}$$

and then to solve the systems of equations obtained by considering respectively those terms independent of R_H and those terms of the first order in R_H. The solution is found only in the case of the symmetric régime (i.e. variations with ϕ and t are ignored), and it is assumed that the ratio of the mean vertical temperature gradient to the mean horizontal temperature gradient, $(\partial\overline{T}/\partial z)/(\partial\overline{T}/\partial r)$, where the bar denotes a value averaged over r and z, is equal to the ratio $\Theta_v/\frac{1}{2}r_o\Theta_H$ of equation (1·12). This leads to a relationship between R_v and R_H of the form

$$\frac{h^2 R_v}{r_0^2} = \tfrac{1}{4}R_H^2\frac{K}{R}\frac{\overline{\tau}_0'^2}{\overline{\tau}_0'}, \tag{5·21}$$

the prime denoting differentiation with respect to r. The function τ_0 (which is found to be a function of r only) must be chosen to conform with the boundary conditions on the temperature distribution. Experimental results are most reliable for the annular case, so in order to make a comparison it is necessary to replace x_{sm} of equation (5·17) and (5·19) by $r_o\beta_{sm}$, where $\beta=\beta_{sm}$ is the sth zero of the equation

$$J_m(\beta r_o)\,Y_m(\beta r_i) - J_m(\beta r_i)\,Y_m(\beta r_o) = 0, \tag{5·22}$$

r_o, r_i being the radii of the outer and inner cylinders. The first zero, β_{1m}, is of vital interest, and this is given to a sufficient degree of accuracy by the asymptotic expansion

$$\beta_{1m} = \frac{\pi}{r_o-r_i}\left\{1+\frac{(4m^2-1)\,(r_o-r_i)^2}{8\pi^2 r_o r_i}+\dots\right\}. \tag{5·23}$$

The function τ_0, which must satisfy the conditions

$$\tau_0'(1) = \tau_0'\left(\frac{r_i}{r_o}\right) = 0 \quad\text{and}\quad \tau_0(1)-\tau_0\left(\frac{r_i}{r_o}\right) = 1$$

was chosen by Davies to satisfy

$$\tau_0' = \frac{12(r-x)\,(1-r^2)}{(1-x)^3\,(3+x)}, \tag{5·24}$$

where $x = r_i/r_o$, and this choice leads to critical values of R_H as follows:

wave-number m	0	1	2	3	4	5
crit. value of R_H	0·19	0·16	0·15	0·12	0·11	0·09

Use of the more refined stability criterion (5·19) leads to results for the critical value of R_H some 5 to 10 % lower than these, so that the large discrepancy with observed results remarked on by Davies does not appear to be substantially reduced when viscosity is taken into consideration. Perhaps the most obvious explanation for this discrepancy lies in the fact that the relation between R_v and R_H obtained by Davies is strictly valid only for the régime of symmetric motion. Moreover, investigations by the present author of later terms in the expansions (5·20) suggest that the convergence of these series solutions is very slow for values of R_H near the observed critical value. It appears then that before any better quantitative agreement may reasonably be expected a more reliable equation relating R_v and R_H must be deduced. Such a relation would be most welcome for the régime of wave-like motion, but the difficulties in this case are considerable, and have so far defied any general analysis. However, the widely held beliefs in the importance of mean isothermal slopes in stability problems of this kind should stimulate further attacks on the problem.

A reason which is less apparent but perhaps of more significance is the occurrence in several of the solutions (3·18) and (4·3) of a factor $\{F-[1/(\mathscr{P}-1)]\}^{-1}$. This factor becomes infinite where $F = 1/(\mathscr{P}-1)$, i.e. when

$$\zeta = \frac{1}{mR_H}\left[\frac{1}{\mathscr{P}-1}-f\right] = \zeta_0, \quad \text{say.} \tag{5·25}$$

If this value of ζ coincides with one of the boundaries of the liquid then the simplification of the determinant of consistency by the method of § 4 is certainly not possible, and the stability criterion (5·19) could not be expected to be true. Moreover, as is shown in appendix B, in passing through the point ζ_0, certain of the solutions (3·18) and (4·3) interlock with each other, so the validity of the simplification of the determinant, and hence the criterion (5·19) is in some doubt for all values of R_H for which $0 \leqslant \zeta_0 \leqslant 1$. Since f is usually a small negative quantity, the first appearance of ζ_0 within this range as R_H increases will be at the boundary $\zeta = 1$. The value of R_H for which $\zeta_0 = 1$ may be deduced provided that a value is allotted to f, and, since no such values are to hand for the experimental results of Fultz quoted by Davies (1956), values may be deduced from the results of Hide (1958) for exactly similar experiments.

In fact
$$f = \frac{\sigma}{2\Omega} = -\frac{m\omega}{2\Omega},$$

where ω = velocity of drift of waves relative to the cylinder. Using Hide's results for an annular cylinder of inner radius 2·35 cm and outer radius 4·85 cm (cf. Fultz's inner and outer radii of 2·45 and 4·90 cm) we obtain

m	1	2	3	4	5
f	no observation	−0·033	−0·037	−0·026	−0·017

Thus the values of R_H above which ζ_0 lies in the range $0 \leqslant \zeta_0 \leqslant 1$ are, assuming $\mathscr{P} = 7$

wave-number m	2	3	4	5
crit. value of R_H	0·10	0·07	0·05	0·035

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

These values of R_H are therefore considerably less than the critical values of R_H obtained above from equation (5·19) and are very close to Fultz's observed critical values for change-over in flow type quoted by Davies, namely

wave-number m	0	1	2	3	4	5
observed critical value of R_H (after Fultz)	0·17	0·15	0·13	0·07	0·05	0·03

This suggests that the reason for change-over may in some way be connected with the appearance within the limits of the fluid of the singular point ζ_0, near which some of the viscous solutions become large, but, in the absence of any results for critical change-over values for variable Prandtl number it seems impossible to draw any definite conclusion in this direction.

One further result remains to be mentioned. Equation (5·19) is satisfied by a second positive value of R_H, a very small value which may be obtained approximately by equating the right-hand side of the equation to zero, i.e.

$$\frac{y^2 \coth^2 y}{16(0{\cdot}214 m R_H)^2 R} - 2{\cdot}4 = 0. \tag{5·26}$$

Using (5·17) and choosing for R a value typical of the experiments, $R = 300$, we obtain values of R_H as follows:

wave-number m	1	2	3	4	5
critical value of R_H	0·04	0·02	0·015	0·01	0·008

Thus there appears to be a second critical value of R_H, below which wave motion is impossible, and this second critical value has in fact been observed by Fultz. The dependence on R of this lower critical value suggests that the reason for its existence is viscosity; below the critical value, wave motion is damped out by the viscosity of the liquid.

The foregoing theory has therefore predicted the existence of a régime of wave-like motion corresponding to a certain range of values of the Rossby number, between what we may term the upper and lower régimes of symmetric flow, and in this respect the results are in qualitative agreement with the experimental evidence of Fultz and theory of Kuo (1956*b*). However, some uncertainty exists as to whether the upper range of values of critical R_H should be chosen to be the range obtained by equating to zero the imaginary part of f, or that obtained by putting $\zeta_0 = 1$. The former method leads to results substantially the same as those of Davies, while the latter certainly gives a much more accurate quantitative agreement with Fultz's experimental figures. Perhaps further conjecture should await the appearance of more detailed experimental evidence, preferably relating to liquids of differing Prandtl number and coefficient of viscosity.

Further discussion of the physical aspects of the problem is beyond the scope of the present paper. It remains to the author to express his thanks to Professor T. V. Davies for suggesting the work contained in this paper and for his constant help and advice, and to Professor T. G. Cowling, F.R.S. for many helpful suggestions and criticisms.

Appendix A. Investigation into the choice of path of integration in the solutions $\overline{W}_5$ and $\overline{W}_6$ of (3·18)

Consider again the equation (2·22) for $\overline{W}$, i.e.

$$-\frac{\mathrm{i}}{R\mathscr{P}}F\mathrm{D}^8\overline{W}+\frac{3\mathrm{i}mR_H}{R\mathscr{P}}\mathrm{D}^7\overline{W}-2F^2\left(1+\frac{2}{\mathscr{P}}\right)\mathrm{D}^6\overline{W}+4FmR_H\left(1-\frac{1}{\mathscr{P}}\right)\mathrm{D}^5\overline{W}$$

$$+4\mathrm{i}F\left\{2F^2-\frac{1-F^2}{\mathscr{P}}\right\}R\mathrm{D}^4\overline{W}+12\mathrm{i}mR_H(1+F^2)\frac{R}{\mathscr{P}}\mathrm{D}^3\overline{W}-8F^2(1-F^2)R^2\mathrm{D}^2\overline{W}$$

$$+16FmR_HR^2\mathrm{D}\overline{W}+8F^2R^2a^2(R_v-F^2)\overline{W}=0. \qquad \text{(A 1)}$$

The two solutions $\overline{W}_7$ and $\overline{W}_8$ arise from letting $R\to\infty$ and considering only the last three terms of (A 1); the other six solutions come from the first seven terms. It is in the two solutions $\overline{W}_5$ and $\overline{W}_6$ that difficulties occur in the determination of the correct path of integration around the branch point at $\zeta=\zeta_1$, and we accordingly seek a representation of these solutions which is valid for small values of $\zeta-\zeta_1$. To do this, a 'change of scale' is made by introducing a new independent variable, χ, such that $\chi=(\zeta-\zeta_1)/\epsilon$ where ϵ is a small parameter. In choosing ϵ, it must be borne in mind that the solutions $\overline{W}_5$ and $\overline{W}_6$, on which attention is to be focused, may be treated as arising effectively from the derivatives of second and fourth order. Thus it follows that the required choice is $\epsilon=R^{-\frac{1}{3}}$; F is then equal to $mR_H\epsilon\chi$, so writing

$$\overline{W}={}_0\overline{W}(\chi)+\epsilon\,{}_1\overline{W}(\chi)+\ldots \qquad \text{(A 2)}$$

and equating to zero the lowest power of ϵ, we have the fourth-order differential equation

$$-\frac{4\mathrm{i}\chi}{\mathscr{P}}\frac{\mathrm{d}^4{}_0\overline{W}}{\mathrm{d}\chi^4}+\frac{12\mathrm{i}}{\mathscr{P}}\frac{\mathrm{d}^3{}_0\overline{W}}{\mathrm{d}\chi^3}-8mR_H\chi^2\frac{\mathrm{d}^2{}_0\overline{W}}{\mathrm{d}\chi^2}+16mR_H\chi\frac{\mathrm{d}\,{}_0\overline{W}}{\mathrm{d}\chi}=0,$$

which may be written in the form

$$\chi^4\mathrm{D}\left[\frac{1}{\chi^3}(\mathrm{D}^2-2\mathrm{i}mR_H\mathscr{P}\chi)\,\mathrm{D}\,{}_0\overline{W}\right]=0, \qquad \text{(A 3)}$$

where $\mathrm{d}/\mathrm{d}\chi$ is written as D.

The solution of equation (A 3) takes the form

$${}_0\overline{W}=A_1\,{}_0\overline{W}_1+A_2\,{}_0\overline{W}_2+A_3\,{}_0\overline{W}_3+A_4\,{}_0\overline{W}_4, \qquad \text{(A 4)}$$

where ${}_0\overline{W}_1$ is a constant, $\mathrm{D}\,{}_0\overline{W}_2$ is a particular integral of the equation (A 5)

$$(\mathrm{D}^2-2\mathrm{i}mR_H\mathscr{P}\chi)\,\Phi=\chi^3, \qquad \text{(A 6)}$$

and $\mathrm{D}\,{}_0\overline{W}_3$, $\mathrm{D}\,{}_0\overline{W}_4$ are two independent solutions of the equation

$$(\mathrm{D}^2-2\mathrm{i}mR_H\chi)\,\Phi=0. \qquad \text{(A 7)}$$

Equation (A 7) can be shown to have the solutions

$$\left.\begin{aligned}\Phi_1&=\chi^{\frac{1}{2}}J_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\},\\ \Phi_2&=\chi^{\frac{1}{2}}Y_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\}.\end{aligned}\right\} \qquad \text{(A 8)}$$

Also the Hankel functions, $H_v^{(1)}$ and $H_v^{(2)}$ are defined as

$$H_v^{(1)}=J_\nu+\mathrm{i}Y_\nu,$$

$$H_v^{(2)}+J_\nu-\mathrm{i}Y_\nu.$$

so it follows that the third and fourth solutions of equation (A 3) are

$$\left.\begin{aligned} {}_0W_3 &= \int \chi^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\}\,\mathrm{d}\chi, \\ {}_0W_4 &= \int \chi^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\}\,\mathrm{d}\chi. \end{aligned}\right\} \tag{A 9}$$

Moreover, the asymptotic expansions of the Hankel functions $H^{(1)}_{\frac{1}{3}}(\omega)$ and $H^{(2)}_{\frac{1}{3}}(\omega)$ are

$$H^{(1)}_{\frac{1}{3}}(\omega) \sim \left(\frac{2}{\pi\omega}\right)^{\frac{1}{2}} \exp\left\{\mathrm{i}\left(\omega-\frac{5\pi}{12}\right)\right\}\left\{1+\sum_{r=1}^{\infty}\frac{(-1)^r\,(\frac{1}{3},r)}{(2\mathrm{i}\omega)^r}\right\} \tag{A 10}$$

valid for $-\pi < \arg\omega < 2\pi$, and

$$H^{(2)}_{\frac{1}{3}}(\omega) \sim \left(\frac{2}{\pi\omega}\right)^{\frac{1}{2}} \exp\left\{-\mathrm{i}\left(\omega-\frac{5\pi}{12}\right)\right\}\left\{1+\sum_{r=1}^{\infty}\frac{(\frac{1}{3},r)}{(2\mathrm{i}\omega)^r}\right\}$$

for $-2\pi < \arg\omega < \pi$.

So if we now put $\omega = \frac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}$, then (A 10) become

$$H^{(1)}_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\} \sim \text{const.}\times\chi^{-\frac{3}{4}}\exp\,[\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{5}{4}\pi\mathrm{i}}\}-\tfrac{5}{12}\pi\mathrm{i}]\times\{1+O(\chi^{-\frac{3}{2}})\}$$

for $-\frac{7}{6}\pi < \arg\chi < \frac{5}{6}\pi$;

$$H^{(2)}_{\frac{1}{3}}\{\tfrac{2}{3}\surd[-2\mathrm{i}mR_H\mathscr{P}]\,\chi^{\frac{3}{2}}\} \sim \text{const.}\times\chi^{-\frac{3}{4}}\exp\,[\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\}+\tfrac{5}{12}\pi\mathrm{i}]\times\{1+O(\chi^{-\frac{3}{2}})\} \tag{A 11}$$

for $-\frac{11}{6}\pi < \arg\chi < \frac{1}{6}\pi$.

Thus it follows that

$$\left.\begin{aligned} {}_0W_3 &\sim \text{const.}\times\int \chi^{-\frac{1}{4}}\exp\,[\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{5}{4}\pi\mathrm{i}}\}]\,\{1+O(\chi^{-\frac{3}{2}})\}\,\mathrm{d}\chi, \\ {}_0W_4 &\sim \text{const.}\times\int \chi^{-\frac{1}{4}}\exp\,[\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\}]\,\{1+O(\chi^{-\frac{3}{2}})\}\,\mathrm{d}\chi, \end{aligned}\right\} \tag{A 12}$$

the expressions being valid in the regions defined in (A 11). If we next take the legitimate step of integrating by parts, and choose our lower limits to be $+\infty$ in ${}_0W_3$ and $-\infty$ in ${}_0W_4$ (since the real part of $\mathrm{e}^{\frac{5}{4}\pi\mathrm{i}}$ is negative, while the real part of $\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ is positive), we get

$${}_0W_3 \sim \text{const.}\times\chi^{-\frac{3}{4}}\exp\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{5}{4}\pi\mathrm{i}}\}+\text{terms of order }(\chi^{-\frac{5}{4}}\exp\{\ \}),$$

$${}_0W_4 \sim \text{const.}\times\chi^{-\frac{3}{4}}\exp\{\tfrac{2}{3}\surd(2mR_H\mathscr{P})\,\chi^{\frac{3}{2}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\}+\text{terms of order }(\chi^{-\frac{5}{4}}\exp\{\ \}),$$

which become, in the notation of (3·18)

$$\left.\begin{aligned} {}_0W_3 &\sim \text{const.}\times(\zeta-\zeta_1)^{-\frac{3}{4}}\exp\left\{-\int_{\zeta_1}^{\zeta}(1+\mathrm{i})\,R^{\frac{1}{2}}\,[mR_H\mathscr{P}(\zeta-\zeta_1)]^{\frac{1}{2}}\,\mathrm{d}\zeta\right\}+\ldots, \\ {}_0W_4 &\sim \text{const.}\times(\zeta-\zeta_1)^{-\frac{3}{4}}\exp\left\{+\int_{\zeta_1}^{\zeta}(1+\mathrm{i})\,R^{\frac{1}{2}}\,[mR_H\mathscr{P}(\zeta-\zeta_1)]^{\frac{1}{2}}\,\mathrm{d}\zeta\right\}+\ldots, \quad \text{as} \quad \chi\to\infty. \end{aligned}\right\} \tag{A 13}$$

These asymptotic expansions for ${}_0W_3$ and ${}_0W_4$ are seen to be in agreement with the solutions W_5 and W_6 of (3·18), and since the region of their validity is well known it is now possible to find a path of integration around the singular point, $\zeta = \zeta_1$, such that the requirements of § 4 are met. Such a path must not cross either of the lines

$$\arg(\zeta-\zeta_1) = \tfrac{1}{6}\pi, \quad \arg(\zeta-\zeta_1) = \tfrac{5}{6}\pi,$$

and so must pass below the point $\zeta = \zeta_1$ in the complex ζ-plane.

Appendix B. Investigation of the behaviour of the asymptotic solutions in passing through the singular point $F = 1/(\mathscr{P}-1)$

In order to discuss the behaviour of the solutions as they pass through the singular point $\zeta = \zeta_0$, at which $F = 1/(\mathscr{P}-1) \equiv F_0$, it is most convenient to return to the basic set of equations (2·17) to (2·21), which become, on replacing $(1+2R_H\zeta)$ by unity, and neglecting $a^2/2R$ and $a^2/2K$ by comparison with F

$$F\bar{\xi}+\bar{\eta} = (2\mathrm{i}R)^{-1}\,\mathrm{D}^2\bar{\xi}, \tag{B 1}$$

$$F\bar{\eta}+\bar{\xi} = -a^2\mathrm{i}\bar{P}+(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\bar{\eta}, \tag{B 2}$$

$$F\bar{W}+\mathrm{i}\bar{\tau} = \mathrm{i}\mathrm{D}\bar{P}+(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\bar{W}, \tag{B 3}$$

$$\bar{\eta}+\mathrm{D}\bar{W} = 0, \tag{B 4}$$

$$F\bar{\tau}-mR_H\bar{P}-\mathrm{i}R_v\bar{W} = (2\mathrm{i}K)^{-1}\,\mathrm{D}^2\bar{\tau}. \tag{B 5}$$

Writing and
$$\left.\begin{aligned} F &= F_0+mR_H\zeta^*,\\ F_0 &= f+mR_H\zeta_0,\end{aligned}\right\}\quad \text{where}\quad \zeta^* = \zeta-\zeta_0 \tag{B 6}$$

and replacing (B 1) and (B 2) by their sum and difference, respectively, we obtain

$$\{1+F_0+mR_H\zeta^*-(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\}\,(\bar{\xi}+\bar{\eta}) = -a^2\mathrm{i}\bar{P}, \tag{B 7}$$

$$\{-1+F_0+mR_H\zeta^*-(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\}\,(\bar{\xi}-\bar{\eta}) = a^2\mathrm{i}\bar{P}, \tag{B 8}$$

$$\{F_0+mR_H\zeta^*-(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\}\,\bar{W}+\mathrm{i}\bar{\tau} = \mathrm{i}\mathrm{D}\bar{P}, \tag{B 9}$$

$$\bar{\eta}+\mathrm{D}\bar{W} = 0, \tag{B 10}$$

$$\left\{1+F_0+mR_H\left(1+\frac{1}{F_0}\right)\zeta^*-(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\right\}\bar{\tau} = \left(1+\frac{1}{F_0}\right)(mR_H\zeta^*\bar{P}+\mathrm{i}R_v\bar{W}). \tag{B 11}$$

Now the solutions of interest are those for which $\bar{\xi}$ and $\bar{\eta}$ are comparable with $\bar{\tau}$ near ζ_0; these are the solutions of suffix 1 and 3 in (4·3), for which $(2\mathrm{i}R)^{-1}\,\mathrm{D}^2 \sim 1+F$, and the solutions of suffix 5 and 6, for which $(2\mathrm{i}R)^{-1}\,\mathrm{D}^2 \sim F\mathscr{P}$. At $F = F_0$, where these solutions have an infinity, $1+F = F\mathscr{P}$, so some interaction between them may be expected.

In equations (B 8) and (B 9) we may replace $(2\mathrm{i}R)^{-1}\,\mathrm{D}^2$ by $1+F_0$ and neglect $mR_H\zeta^*$ for small ζ^*, but the small value of $(2\mathrm{i}R)^{-1}\,\mathrm{D}^2-1-F$ is vital in equations (B 7) and (B 11). Thus we have

$$-2(\bar{\xi}-\bar{\eta}) = a^2\mathrm{i}\bar{P}, \tag{B 12}$$

$$-\bar{W}+\mathrm{i}\bar{\tau} = \mathrm{i}\mathrm{D}\bar{P}, \tag{B 13}$$

$$\{1+F_0+mR_H\zeta^*-(2\mathrm{i}R)^{-1}\,\mathrm{D}^2\}\,(\bar{\xi}+\bar{\eta}) = -a^2\mathrm{i}\bar{P}, \tag{B 14}$$

$$\mathrm{D}\bar{W} = -\bar{\eta} = \tfrac{1}{2}\{(\bar{\xi}-\bar{\eta})-(\bar{\xi}+\bar{\eta})\} = -\tfrac{1}{2}\{\tfrac{1}{2}a^2\mathrm{i}\bar{P}+(\bar{\xi}+\bar{\eta})\}. \tag{B 15}$$

From (B 14) it follows that $a^2\mathrm{i}\bar{P} \ll \bar{\xi}+\bar{\eta}$, so (B 15) approximates to

$$\mathrm{D}\bar{W} = -\tfrac{1}{2}(\bar{\xi}+\bar{\eta}). \tag{B 16}$$

Similarly, from (B 11) $\bar{\tau} \gg mR_H\zeta^*\bar{P}+\mathrm{i}R_v\bar{W}$, so that either $\bar{\tau} \gg \bar{P}$, $\bar{W}$, or $\bar{\tau}$ is not large compared with $\bar{P}$, $\bar{W}$, but $mR_H\zeta^*\bar{P} \doteqdot -\mathrm{i}R_v\bar{W}$. The second alternative is inadmissible since it would imply that in equation (B 13) the right-hand side $\gg$ left-hand side. Thus $\bar{\tau} \gg mR_H\zeta^*\bar{P}+\mathrm{i}R_v\bar{W}$, and equation (B 13) becomes

$$\bar{\tau} = \mathrm{D}\bar{P}. \tag{B 17}$$

Whence, substituting in (B 11)

$$\left\{1+F_0+mR_H\left(1+\frac{1}{F_0}\right)\zeta^*-(2\mathrm{i}R)^{-1}\mathrm{D}^2\right\}\mathrm{D}\bar{P}=\left(1+\frac{1}{F_0}\right)(mR_H\zeta^*\bar{P}+\mathrm{i}R_v\bar{W}) \qquad \text{(B 18)}$$

and combining equations (B 14) and (B 16),

$$\{1+F_0+mR_H\zeta^*-(2\mathrm{i}R)^{-1}\mathrm{D}^2\}\mathrm{D}\bar{W}=\tfrac{1}{2}a^2\mathrm{i}\bar{P}. \qquad \text{(B 19)}$$

Equations (B 18) and (B 19) now determine $\bar{P}$ and $\bar{W}$; since we are interested in solutions for which $(2\mathrm{i}R)^{-1}\mathrm{D}^2-1-F_0$ is small we write

$$\bar{W}=\mathrm{e}^{\gamma\zeta^*}\bar{W}_1, \quad \bar{P}=\mathrm{e}^{\gamma\zeta^*}\bar{P}_1, \quad \text{where} \quad \gamma^2=2\mathrm{i}R(1+F_0), \qquad \text{(B 20)}$$

where the functions $\bar{W}_1$ and $\bar{P}_1$ do not vary rapidly near $\zeta^*=0$. Then

$$F_0\left(\frac{\mathrm{D}^3}{\gamma^2}+\frac{3\mathrm{D}^2}{\gamma}\right)\bar{P}_1-(mR_H\zeta^*-2F_0)\,\mathrm{D}\bar{P}_1+mR_H(1-\gamma\zeta^*)\,\bar{P}_1=-\mathrm{i}R_v\bar{W}_1, \qquad \text{(B 21)}$$

$$(1+F_0)\left(\frac{\mathrm{D}^3}{\gamma^2}+\frac{3\mathrm{D}^2}{\gamma}\right)\bar{W}_1-\{mR_H\zeta^*-2(1+F_0)\}\,\mathrm{D}\bar{W}_1-\gamma mR_H\zeta^*\bar{W}_1=-\tfrac{1}{2}a^2\mathrm{i}\bar{P}_1. \qquad \text{(B 22)}$$

Since $\bar{W}_1$, $\bar{P}_1$ do not vary rapidly near $\zeta^*=0$, it may be assumed that

$$\left.\begin{array}{l}\dfrac{\mathrm{D}\bar{P}_1}{\bar{P}_1},\quad \dfrac{\mathrm{D}^2\bar{P}_1}{\mathrm{D}\bar{P}_1},\quad \dfrac{\mathrm{D}^3\bar{P}_1}{\mathrm{D}^2\bar{P}_1}\ll\gamma\\[2ex] \dfrac{\mathrm{D}\bar{W}_1}{\bar{W}_1},\quad \dfrac{\mathrm{D}^2\bar{W}_1}{\mathrm{D}\bar{W}_1},\quad \dfrac{\mathrm{D}^3\bar{W}_1}{\mathrm{D}^2\bar{W}_1}\ll\gamma\end{array}\right\} \qquad \text{(B 23)}$$

and also that, because of the smallness of ζ^*, $\gamma mR_H\zeta^*\bar{P}_1$ is not large compared with $\mathrm{D}\bar{P}_1$.

Equations (B 21) and (B 22) then simplify to

$$2F_0\mathrm{D}\bar{P}_1+mR_H(1-\gamma\zeta^*)\,\bar{P}_1=-\mathrm{i}R_v\bar{W}_1, \qquad \text{(B 24)}$$

$$2(1+F_0)\,\mathrm{D}\bar{W}_1-\gamma mR_H\zeta^*\bar{W}_1=-\tfrac{1}{2}a^2\mathrm{i}\bar{P}_1. \qquad \text{(B 25)}$$

For large values of $|\gamma\zeta^*|$ the solutions of these equations should be in agreement with the solutions (4·3); in fact solutions are of two types:

(i) those in which $\bar{P}_1\gg\bar{W}_1$, given by

$$\frac{\mathrm{D}\bar{P}_1}{\bar{P}_1}\sim\frac{\gamma\zeta^*mR_H}{2F_0}, \quad \text{i.e.} \quad \bar{P}_1\propto\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\} \quad \text{and} \quad \bar{W}_1\propto\frac{1}{\gamma\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\};$$

(ii) those in which $\bar{W}_1\gg\bar{P}_1$, given by

$$\frac{\mathrm{D}\bar{W}_1}{\bar{W}_1}\sim\frac{\gamma\zeta^*mR_H}{2(1+F_0)}, \quad \text{i.e.} \quad \bar{W}_1\propto\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\} \quad \text{and} \quad \bar{P}_1\propto\frac{1}{\gamma\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\}.$$

These solutions agree with the results of § 3, since

$$\exp\left\{\gamma\zeta^*+\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\}=\exp(R^{\frac{1}{2}}\Psi),$$

where
$$R^{\frac{1}{2}}\Psi'=\gamma\left(1+\frac{\zeta^*mR_H}{2F_0}\right);$$

therefore
$$\Psi'^2=\frac{\gamma^2}{R}\left\{1+\frac{\zeta^*mR_H}{F_0}+O(\zeta^{*2})\right\}\doteqdot\frac{\gamma^2}{R}\frac{F}{F_0}=2\mathrm{i}F\mathscr{P},$$

i.e.
$$\Psi'^2=(Q_5')^2=(Q_6')^2$$

and therefore the solutions (i) above correspond to solutions with suffix 5 and 6 of § 3; similarly solutions (ii) correspond to solutions with suffix 1 and 3 of § 3.

We now wish to find out whether the solutions of type (i) and (ii) above are valid for ζ^* both positive and negative, or whether they change in passing through $\zeta^* = 0$.

Suppose first that the real part of $\gamma m R_H$ is positive, so that the exponential in solution (i) is asymptotically large compared with that in solution (ii). Then we assume that a solution exists such that, for both positive and negative ζ^*,

$$\bar{P}_1 \doteqdot A \exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0}\right\}. \tag{B 26}$$

In order to check whether or not this assumption is self-consistent, we must substitute (B 26) in (B 25), and solve for $\overline{W}_1$, leading to

$$2(1+F_0)\,\mathrm{D}\overline{W}_1 - \gamma\zeta^* m R_H \overline{W}_1 = -\tfrac{1}{2}Aa^2\mathrm{i}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0}\right\}, \tag{B 27}$$

i.e. $$\frac{\mathrm{d}}{\mathrm{d}\zeta^*}\left[\overline{W}_1 \exp\left\{-\frac{\gamma\zeta^{*2} m R_H}{4(1+F_0)}\right\}\right] = -\frac{Aa^2\mathrm{i}}{4(1+F_0)}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}.$$

Thus $$\overline{W}_1 \exp\left\{-\frac{\gamma\zeta^{*2} m R_H}{4(1+F_0)}\right\} = -\frac{Aa^2\mathrm{i}}{4(1+F_0)}\int \exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^*$$

and since, for consistency, we must have

$$\overline{W}_1 \sim \frac{1}{\gamma\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0}\right\},$$

it follows that the limits of integration on the right-hand side should be 0 and ζ^*. This is because

$$\int_a^{\zeta^*} \exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^* = \left[\frac{2F_0(1+F_0)}{\gamma\zeta^* m R_H}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}\right]_a^{\zeta^*} + \int_a^{\zeta^*}\frac{2F_0(1+F_0)}{\gamma\zeta^{*2} m R_H}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^*,$$

so by taking $|\gamma\zeta^{*2}| \gg |\gamma a^2| \gg 1$, we have

$$\int_0^{\zeta^*} \exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^* \sim \frac{2F_0(1+F_0)}{\gamma\zeta^* m R_H}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0(1+F_0)}\right\},$$

giving $$\overline{W}_1 \sim -\frac{Aa^2\mathrm{i}F_0}{2\gamma\zeta^* m R_H}\exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4F_0}\right\} \tag{B 28}$$

for positive and negative ζ^*.

Also the solution for $\overline{W}_1$ remains finite at $\zeta^* = 0$. It appears then that the term on the right-hand side of equation (B 24) does not appreciably affect the solution (B 26) in passing through $\zeta^* = 0$, and therefore the basic assumption that (B 26) is a solution for positive and negative ζ^* is self-consistent.

Suppose next that a solution of type (ii) exists, so that for both positive and negative ζ^*,

$$\overline{W}_1 \doteqdot B \exp\left\{\frac{\gamma\zeta^{*2} m R_H}{4(1+F_0)}\right\}. \tag{B 29}$$

In order to check for self-consistency, we substitute (B 29) in (B 24) and solve for $\bar{P}_1$, leading to

$$2F_0 D\bar{P}_1 + (1-\gamma\zeta^*)\, mR_H \bar{P}_1 = -\mathrm{i}BR_v \exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\}, \tag{B 30}$$

i.e.
$$\frac{\mathrm{d}}{\mathrm{d}\zeta^*}\left[\bar{P}_1 \exp\left\{\frac{(2\zeta^*-\gamma\zeta^{*2})\, mR_H}{4F_0}\right\}\right] = -\frac{\mathrm{i}BR_v}{2F_0}\exp\left\{\frac{2\zeta^* mR_H}{4F_0} - \frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\}.$$

Thus
$$\bar{P}_1 \exp\left\{\frac{(2\zeta^*-\gamma\zeta^{*2})\, mR_H}{4F_0}\right\} = \frac{\mathrm{i}BR_v}{2F_0}\int \exp\left\{\frac{2\zeta^* mR_H}{4F_0} - \frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^*$$

and, for consistency, we must have

$$\bar{P}_1 \sim \frac{1}{\gamma\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\}.$$

For $\zeta^* > 0$ this condition can be satisfied by taking the limits of integration to be ζ^* and ∞; then the integral converges because the real part of γmR_H is positive, and for large $\gamma\zeta^*$ it is approximately equal to

$$\frac{2F_0(1+F_0)}{\gamma\zeta^* mR_H}\exp\left\{\frac{2\zeta^* mR_H}{4F_0} - \frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\},$$

which gives
$$\bar{P}_1 \sim \frac{\mathrm{i}R_v B(1+F_0)}{\gamma\zeta^* mR_H}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\}. \tag{B 31}$$

For $\zeta^* < 0$, however, with these limits of integration, the integral becomes approximately

$$\frac{2F_0(1+F_0)}{\gamma\zeta^* mR_H}\exp\left\{\frac{2\zeta^* mR_H}{4F_0} - \frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\} + \left\{\frac{4\pi F_0(1+F_0)}{\gamma mR_H}\right\}^{\frac{1}{2}}\exp\left\{\frac{mR_H(1+F_0)}{4F_0\gamma}\right\},$$

which gives
$$\bar{P}_1 \sim \mathrm{i}R_v B\left[\frac{1+F_0}{\gamma\zeta^* mR_H}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\} + \left\{\frac{\pi(1+F_0)}{\gamma F_0 mR_H}\right\}^{\frac{1}{2}}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4F_0}\right\}\right] \tag{B 32}$$

and self-consistency is violated because the extra term in $\bar{P}_1$ introduces an extra term in $\bar{W}_1$ for $\zeta^* < 0$, which is contrary to our original supposition that

$$\bar{W}_1 = B\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\} \tag{B 29}$$

for positive and negative ζ^*.

We must accordingly assume that $\bar{W}_1$ is given by

$$\bar{W}_1 \doteqdot \exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4(1+F_0)}\right\}\left[B + \frac{R_v Ba^2}{4(1+F_0)}\left\{\frac{\pi(1+F_0)}{\gamma F_0 mR_H}\right\}^{\frac{1}{2}}\int_0^{\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^*\right]. \tag{B 33}$$

The extra term in $\bar{W}_1$ will of course introduce an extra term in $\bar{P}_1$, but this may be shown to be negligible. Substituting (B 33) in (B 24) we find that the extra term is

$$\exp\left\{\frac{-(2\zeta^*-\gamma\zeta^{*2})\, mR_H}{4F_0}\right\}\frac{\mathrm{i}R_v}{2F_0}\int_{\zeta^*}^{\infty}\frac{R_v Ba^2}{4(1+F_0)}\left\{\frac{\pi(1+F_0)}{\gamma F_0 mR_H}\right\}^{\frac{1}{2}}\left[\int_0^{\zeta^*}\exp\left\{\frac{\gamma\zeta^{*2} mR_H}{4F_0(1+F_0)}\right\}\mathrm{d}\zeta^*\right]$$
$$\times\exp\left\{\frac{(2\zeta^*-\gamma\zeta^{*2})\, mR_H}{4F_0}\right\}\mathrm{d}\zeta^*. \tag{B 34}$$

For large $\gamma\zeta^{*2}$ the inner integral

$$\int_0^{\zeta^*} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0(1+F_0)}\right\} \mathrm{d}\zeta^*,$$

is comparable with

$$\frac{2F_0(1+F_0)}{\gamma\zeta^* mR_H} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0(1+F_0)}\right\}$$

and, as $\zeta^* \to 0$,

$$\int_0^{\zeta^*} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0(1+F_0)}\right\} \mathrm{d}\zeta^* \to 0.$$

Hence the term (B 34) is comparable with

$$\exp\left\{\frac{-(2\zeta^*-\gamma\zeta^{*2})\,mR_H}{4F_0}\right\} \frac{\mathrm{i}R_v Ba^2}{4\gamma mR_H} \left\{\frac{\pi(1+F_0)}{\gamma F_0 mR_H}\right\}^{\frac{1}{2}} \int_{\zeta^*}^{\infty} \frac{1}{\zeta^*} \exp\left\{\frac{2\zeta^* mR_H}{4F_0} - \frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\} \mathrm{d}\zeta^*,$$

where no account need be taken of the infinity of $1/\zeta^*$. This is certainly small, of order $1/\gamma$, compared with the terms retained in the expression (B 32) for $\bar{P}_1$.

Thus, for large negative ζ^*, the solutions of type (ii) become

$$\left.\begin{aligned} \bar{P}_1 &= \mathrm{i}R_v B\left[\frac{1+F_0}{\gamma\zeta^* mR_H} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\} + \left\{\frac{\pi(1+F_0)}{\gamma F_0 mR_H}\right\}^{\frac{1}{2}} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\}\right], \\ \bar{W}_1 &= B\left[\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\} + \frac{R_v a^2}{2\gamma\zeta^* mR_H}\left\{\frac{\pi(1+F_0)}{\gamma mR_H}\right\}^{\frac{1}{2}} \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\}\right]. \end{aligned}\right\} \quad \text{(B 35)}$$

The reason why the solutions of type (ii) appear to interlock with those of type (i), while those of type (i) show no converse effect, is that, when $\gamma\zeta^{*2}$ is large,

$$\exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4F_0}\right\} \gg \exp\left\{\frac{\gamma\zeta^{*2}mR_H}{4(1+F_0)}\right\}$$

and therefore a small contamination of a solution of type (i) by one of type (ii) is not noticeable; a contamination in the reverse sense, which occurs here, is important.

This analysis indicates that some of the asymptotic solutions obtained in § 3 do not continue unchanged through the singular point $\zeta = \zeta_0$, but interlock with each other. If this is so, then the validity of the determinant of coefficients (4·4) is suspect for values of R_H such that $0 \leqslant \zeta_0 \leqslant 1$, and, in order to obtain the correct form, the exact transformations of the asymptotic integrals in passing through $\zeta = \zeta_0$ would have to be found; the analytical difficulties of this problem put it beyond the scope of the present paper.

References

Davies, T. V. 1953 *Phil. Trans.* A, **246**, 81.
Davies, T. V. 1956 *Phil. Trans.* A, **249**, 27.
Davies, T. V. 1959 *J. Fluid Mech.* **5**, 593.
Fultz, D. 1956 *Rep. Hydrodynamic Lab.* (*Univ. of Chicago*), no. AFCRC-TR 56–259.
Fultz, D. & Riehl, H. 1957 *Quart. J. R. Met. Soc.* **83**, 215.
Hide, R. 1958 *Phil. Trans.* A, **250**, 441.
Kuo, H. L. 1954 *J. Met.* **11**, 399.
Kuo, H. L. 1955 *J. Mar. Res.* **14**, 14.
Kuo, H. L. 1956*a* *J. Met.* **13**, 81.
Kuo, H. L. 1956*b* *J. Met.* **13**, 521.
Lin, C. C. 1955 *The theory of hydrodynamic stability.* Cambridge University Press.
Lorenz, E. W. 1954 *Final Rep., Gen. Circ. Project, Mass. Inst. Tech.*

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF